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Surfaces with constant extrinsically Gaussian curvature in the Heisenberg group

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Abstract

In this work we study constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group which are invariant under the 1-parameter groups of isometries.

Keywords: Constant extrinsically Gaussian curvature Surfaces, Homogeneous group.

MSC: 49Q20 53C22.

1. Introduction

In 1982, W. P. Thurston formulated a geometric conjecture for three dimensional manifolds, namely every compact orientable 3-manifold admits a canonical decomposition into pieces, each of them having a canonical geometric structure from the following eight maximal and simply connected homogenous Riemannian spaces: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $SL(2,\mathbb{R})$, $\mathbb{H}_3$ and Sol_3. See e.g. [34].

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in three homogenous spaces focusing on flat and constant Gaussian curvature surfaces. Many works are studying the geometry of surfaces in homogeneous 3-manifolds. See for example [2–4, 9, 12, 14–16, 21, 22, 24, 36].

The concept of translation surfaces in $\mathbb{R}^3$ can be generalized the surfaces in the three dimensional Lie group, in particular, homogeneous manifolds. In Euclidean

3-space, every cylinder is flat. Conversely, complete flat surfaces in $\mathbb{E}^3$ are cylinders over complete curves. See [20]. López and Munteanu [17] studied invariant surfaces with constant mean curvature and constant Gaussian curvature in Sol_3 space. Yoon and Lee [37] studied translation surfaces in Heisenberg group $\mathbb{H}_3$ whose position vector x satisfies the equation $\Delta x = Ax$, where Δ is the Laplacian operator of the surface and A is a 3×3-real matrix.

Flat G_4-invariant surfaces are nothing but surfaces invariant under $SO(2)$-action, i.e. rotational surfaces. Flat rotational surfaces are classified by Caddeo, Piu and Ratto in [8].

In [14], J. I. Inoguchi give a classification of intrinsically flat G_1-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$. Let M be a surface invariant under G_3, then M is locally expressed as

$$X(u, v) = (0, 0, v).(x(u), y(u), 0) = (x(u), y(u), v), \ u \in I, \ v \in \mathbb{R}.$$

Here I is an open interval and u is the arclength parameter. Note that $(x, y, 0)$ and $(0, 0, v)$ commute. Then the sectional curvature $K(X_x \wedge X_y) = \frac{1}{4}$ and the extrinsically Gaussian curvature $K_{ext} = -\frac{1}{4}$. Direct computation show that M is flat. (cf. [12–14, 28]).

The paper is divided according the type of surfaces invariant under 1-parameter subgroups of isometries $\{G_i\}_{i=1,2,3,4}$. So, in section 3 we classify G_1-invariant surfaces of the Heisenberg group $\mathbb{H}_3$ with constant extrinsically Gaussian curvature K_{ext}, including extrinsically flat G_1-invarinant surfaces.

In section 4 we classify G_2-invariant surfaces of the Heisenberg group $\mathbb{H}_3$ with constant extrinsically Gaussian curvature K_{ext}, including extrinsically flat G_2-invariant surfaces.

2. Preliminaries

The 3-dimensional Heisenberg group $\mathbb{H}_3$ is the simply connected and connected 2-step nilpotent Lie group. Which has the following standard representation in $GL(3, \mathbb{R})$

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$$

with $r, s, t \in \mathbb{R}$. The Lie algebra $\mathfrak{h}_3$ of $\mathbb{H}_3$ is given by the matrices

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$.The exponential map $exp : \mathfrak{h}_3 \to \mathbb{H}_3$ is a global diffeomorphism, and is given by

$$\exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The Heisenberg group $\mathbb{H}_3$ is represented as the cartesian 3-space $\mathbb{R}^3(x, y, z)$ with group structure:

$$(x_1, y_1, z_1).(x_2, y_2, z_2) := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1 \right).$$

We equip $\mathbb{H}_3$ with the following left invariant Riemannian metric

$$g := dx^2 + dy^2 + \left(dz + \frac{1}{2}(ydx - xdy) \right)^2.$$

The identity component $I^\circ(\mathbb{H}_3)$ of the full isometry group of $(\mathbb{H}_3, g)$ is the semi-direct product $SO(2) \ltimes \mathbb{H}_3$. The action of $SO(2) \ltimes \mathbb{H}_3$ is given explicitly by

$$\begin{aligned} A &= \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) . \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \frac{1}{2}(a\sin\theta - b\cos\theta) & \frac{1}{2}(a\cos\theta + b\sin\theta) & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned}$$

In particular, rotational around the z-axis and translations:

$$(x, y, z) \to (x, y, z + a), a \in \mathbb{R}$$

along the z-axis are isometries of $\mathbb{H}_3$.

The Lie algebra $\mathfrak{h}_3$ of $I^\circ(\mathbb{H}_3)$ is generated by the following Killing vector fields:

$$F_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z},$$

$$F_3 = \frac{\partial}{\partial z}, \quad F_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

One can check that F_1, F_2, F_3 are infinitesimal transformations of the 1-parameter groups of isometries defined by

$$G_1 = \{(t, 0, 0) | t \in \mathbb{R}\}, \ G_2 = \{(0, t, 0) | t \in \mathbb{R}\}, \ G_3 = \{(0, 0, t) | t \in \mathbb{R}\},$$

respectively. Here this groups acts on $\mathbb{H}_3$ by the left translation. The vector field F_4 generates the group of rotations around the z-axis. Thus G_4 is identified with $SO(2)$.

Definition 2.1. A surface Σ in the Heisenberg space $\mathbb{H}_3$ is said to be invariant surface if it is invariant under the action of the 1-parameter subgroups of isometries $\{G_i\}$, with $i \in \{1,2,3,4\}$.

The Lie algebra $\mathfrak{h}_3$ of $\mathbb{H}_3$ has an orthonormal basis $\{E_1, E_2, E_3\}$ defined by

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \ E_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \ E_3 = \frac{\partial}{\partial z}.$$

The Levi-Civita connection ∇ of g,in terms of the basis $\{E_i\}_{i=1,2,3}$ is explicitly given as follows

$$\begin{cases} \nabla_{E_1}E_1 = 0, \nabla_{E_1}E_2 = \frac{1}{2}E_3, \nabla_{E_1}E_3 = -\frac{1}{2}E_2 \\ \nabla_{E_2}E_1 = -\frac{1}{2}E_3, \nabla_{E_2}E_2 = 0, \nabla_{E_2}E_3 = \frac{1}{2}E_1 \\ \nabla_{E_3}E_1 = -\frac{1}{2}E_2, \nabla_{E_3}E_2 = \frac{1}{2}E_1, \nabla_{E_3}E_3 = 0 \end{cases}$$

The Riemannian curvature tensor R is a tensor field on $\mathbb{H}_3$ defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The components $\{R^l_{ijk}\}$ are computed as

$$R^1_{212} = -\frac{3}{4}, \ R^1_{313} = \frac{1}{4}, \ R^2_{323} = \frac{1}{4}.$$

Let us denote $K_{ij} = K(E_i, E_j)$ the sectional curvature of the plane spanned by E_i and E_j.Then we get easily the following:

$$K_{12} = -\frac{3}{4}, \ K_{13} = -\frac{1}{4}, \ K_{23} = -\frac{1}{4}.$$

The Ricci curvature Ric is defined by

$$Ric(X,Y) = trace\{Z \to R(Z,X)Y\}.$$

The components $\{R_{ij}\}$ of Ric are defined by

$$Ric(E_i, E_j) = R_{ij} = \sum_{k=1}^{3} \langle R(E_i, E_k)E_k, E_j \rangle .$$

The components $\{R_{ij}\}$ are computed as

$$R_{11} = -\frac{1}{2}, \ R_{12} = R_{13} = R_{23} = 0, \ \ R_{22} = -\frac{1}{2}, \ R_{33} = \frac{1}{2}.$$

The scalar curvature S of $\mathbb{H}_3$ is constant and we have

$$S = trRic = \sum_{i=1}^{3} Ric(E_i, E_i) = -\frac{1}{2}.$$

3. Constant extrinsically Gaussian curvature G_1-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$

3.1.

In this subsection we study complete extrinsically flat translation surfaces Σ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup G_1. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$. Thus γ is given by $\gamma(y) = (0, y, v(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, 0, 0).(0, y, v(y)) = (x, y, v(y) + \frac{xy}{2}),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_x = (1, 0, \frac{y}{2}) = E_1 + yE_3.$$

and

$$e_2 := X_y = (0, 1, v' + \frac{x}{2}) = E_2 + v'E_3.$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1\rangle = 1 + y^2,\ F = \langle e_1, e_2\rangle = yv',\ G = \langle e_2, e\rangle = 1 + v'^2.$$

As a unit normal field we can take

$$N = \frac{-y}{\sqrt{1 + y^2 + v'^2}} E_1 - \frac{v'}{\sqrt{1 + y^2 + v'^2}} E_2 + \frac{1}{\sqrt{1 + y^2 + v'^2}} E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1} e_1 = -yE_2$$

$$\widetilde{\nabla}_{e_1} e_2 = \frac{y}{2} E_1 - \frac{v'}{2} E_2 + \frac{1}{2} E_3$$

$$\widetilde{\nabla}_{e_2} e_2 = v'E_1 + v''E_3.$$

The coefficients of the second fundamental form are

$$l = \langle \widetilde{\nabla}_{e_1} e_1, N\rangle = \frac{yv'}{\sqrt{1 + y^2 + v'^2}}$$

$$m = \langle \widetilde{\nabla}_{e_1} e_2, N\rangle = \frac{-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}}{\sqrt{1 + y^2 + v'^2}}$$

$$n = \langle \widetilde{\nabla}_{e_2} e_2, N\rangle = \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}.$$

Let K_{ext} be the extrinsic Gauss curvature of Σ,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{-y^2v'^2 + yv'v'' - (-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2})^2}{(1 + y^2 + v'^2)^2}.$$

Thus Σ is extrinsically flat invariant surface in Heisenberg group $\mathbb{H}_3$ if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$-y^2v'^2 + yv'v'' - \left(-\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2 = 0 \tag{3.1}$$

to classify extrinsically flat invariant surfaces must solve the equation (3.1). We can writes equation (3.1) as

$$y^2 + yv'v'' - \left(\frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}\right)^2 = 0 \tag{3.2}$$

we assume that $z = \frac{y^2}{2} + \frac{v'^2}{2} + \frac{1}{2}$. Then

$$\begin{cases} z' = y + v'v'' \\ v'v'' = z' - y \\ v'^2 = 2z - y^2 - 1. \end{cases} \tag{3.3}$$

Therefore equation (3.2) becomes

$$yz' - z^2 = 0. \tag{3.4}$$

equation (3.4) implies that

$$-\frac{z'}{z^2} = -\frac{1}{y}. \tag{3.5}$$

and equation (3.5) implies that

$$z = \frac{1}{-\ln(y) + \alpha}. \tag{3.6}$$

where $\alpha \in \mathbb{R}$, and if $y \neq e^{\alpha}$.

From (3.3) and (3.6), we have

$$\begin{aligned} v'^2 &= 2z - y^2 - 1 \\ &= \frac{2}{-\ln(y) + \alpha} - y^2 - 1. \end{aligned}$$

Thus

$$v' = \sqrt{\frac{2}{-\ln(y) + \alpha} - y^2 - 1}.$$

As conclusion, we have

Theorem 3.1. $\bullet$ *The only non-extendable extrinsically flat translation surfaces in the 3-dimensional Heisenberg group $\mathbb{H}_3$ invariant under the 1-parameter subgroup $G_1 = \{(t,0,0) \in \mathbb{H}_3/t \in \mathbb{R}\}$, are the surfaces whose parametrization is $X(x,y) = \big(x, y, v(y) + \frac{xy}{2}\big)$ where y and v satisfy*

$$v(y) = \int \sqrt{\frac{2}{-\ln(y)+\alpha} - y^2 - 1dy}.$$

where $\alpha \in \mathbb{R}$, and $y \neq e^{\alpha}$.

$\bullet$ *There are no complete extrinsically flat translation surfaces in the 3-dimensional Heisenberg group $\mathbb{H}_3$ invariant under the 1-parameter subgroup $G_1 = \{(t,0,0) \in \mathbb{H}_3/t \in \mathbb{R}\}$.*

Remark 3.2. Let Σ be a G_1-invariant translation surfaces in the 3-dimensional Heisenberg space. Then Σ is locally expressed as

$$X(x,y) = (0, y, v(y)) \,.\, (x, 0, 0) = \left(x, y, v(y) - \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature K_{ext} of Σ is computed as

$$K_{ext} = \frac{\left((v' - x)^2 - 1\right)^2}{4\left(1 + (v' - x)^2\right)^2}.$$

Thus Σ can not be of constant extrinsically Gaussian curvature.

3.2.

In this subsection we study complete constant extrinsically Gaussian curvature translation surfaces Σ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup G_1. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$. Thus γ is given by $\gamma(y) = (0, y, v(y))$. Therefore the generated surface is parameterized by

$$X(x,y) = (x,0,0).(0,y,v(y)) = (x, y, v(y) + \frac{xy}{2}),\ (x,y) \in \mathbb{R}^2.$$

Theorem 3.3. $\bullet$ *The G_1-invariant constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group $\mathbb{H}_3$, are:*

1. $K_{ext} = -\frac{1}{4}$.
   The surfaces of equation

$$z = v(y) + \frac{xy}{2} = \frac{xy}{2} + \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan\left(\frac{y}{\sqrt{\beta - y^2}}\right),$$

where $\beta \in \mathbb{R}$.

2. $K_{ext} \neq -\frac{1}{4}$.
Then y and v satisfy

$$v(y) = \int \sqrt{\frac{1}{-2(K_{ext} + \frac{1}{4})\ln(y) + \gamma} - y^2 - 1dy}.$$

where $\gamma \in \mathbb{R}$, and $y \neq e^{\frac{\gamma}{2(K_{ext}+\frac{1}{4})}}$.

• *There are no complete constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group $\mathbb{H}_3$ invariant under the 1-parameter subgroup G_1.*

Proof. From (4.1) and (3.2) we have

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{y^2 + yv'v'' - \frac{1}{4}\left(1 + y^2 + v'^2\right)^2}{(1 + y^2 + v'^2)^2}. \tag{3.7}$$

1. If $K_{ext} = -\frac{1}{4}$. Then equation (3.7) becomes

$$y^2 + yv'v'' = 0 \tag{3.8}$$

We note that y equal zero is solution of the equation(3.8).
If y is different to zero ($y \neq 0$), equation (3.8) becomes

$$v'v'' = -y.$$

Integration gives us

$$v(y) = \frac{1}{2}y\sqrt{2\beta - y^2} + \arctan\left(\frac{y}{\sqrt{\beta - y^2}}\right),$$

where $\beta \in \mathbb{R}$.

2. If $K_{ext} \neq -\frac{1}{4}$. Then equation (3.7) becomes

$$y^2 + yv'v'' = (K_{ext} + \frac{1}{4})(1 + y^2 + v'^2)^2.$$

In fact, put $z = 1 + y^2 + v'^2$. Then z satisfies

$$\frac{1}{2}yz' = (K_{ext} + \frac{1}{4})z^2.$$

Hence we have

$$z = \frac{1}{-2(K_{ext} + \frac{1}{4})y + \gamma},$$

where $\gamma \in \mathbb{R}$, and $y \neq e^{\frac{\gamma}{2(K_{ext}+\frac{1}{4})}}$. Using the equation $z = 1 + y^2 + v'^2$, we get

$$v'^2 = \frac{1}{-2(K_{ext} + \frac{1}{4})y + \gamma} - y^2 - 1. \qquad \square$$

4. Constant extrinsically Gaussian curvature G_2-invariant translation surfaces in Heisenberg group $\mathbb{H}_3$

In this section we study constant complete extrinsically flat translation surfaces Σ in Heisenberg group $\mathbb{H}_3$ which are invariant under the one parameter subgroup G_2. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{y = 0\}$. Discarding the trivial case of a vertical plane $\{x = x_0\}$. Thus γ is given by $\gamma(x) = (x, 0, f(x))$. Therefore the generated surface is parameterized by

$$X(x, y) = (0, y, 0).(x, 0, f(x)) = (x, y, f(x) - \frac{xy}{2}),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_x = (1, 0, f' - \frac{y}{2}) = E_1 + f'E_3.$$

and

$$e_2 := X_y = (0, 1, -\frac{x}{2}) = E_2 - xE_3.$$

The coefficients of the first fundamental form are:

$$E = \langle e_1, e_1\rangle = 1 + f'^2,\ F = \langle e_1, e_2\rangle = -xf',\ G = \langle e_2, e\rangle = 1 + x^2.$$

As a unit normal field we can take

$$N = \frac{-f'}{\sqrt{1 + x^2 + f'^2}}E_1 + \frac{x}{\sqrt{1 + x^2 + f'^2}}E_2 + \frac{1}{\sqrt{1 + x^2 + f'^2}}E_3.$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1} e_1 = -f'E_2 + f''E_3$$

$$\widetilde{\nabla}_{e_1} e_2 = \frac{f'}{2}E_1 + \frac{x}{2}E_2 - \frac{1}{2}E_3$$

$$\widetilde{\nabla}_{e_2} e_2 = -xE_1.$$

The coefficients of the second fundamental form are

$$l = \langle\widetilde{\nabla}_{e_1} e_1, N\rangle = \frac{-xf' + f''}{\sqrt{1 + x^2 + f'^2}}$$

$$m = \langle\widetilde{\nabla}_{e_1} e_2, N\rangle = \frac{-\frac{f'^2}{2} + \frac{x^2}{2} - \frac{1}{2}}{\sqrt{1 + x^2 + f'^2}}$$

$$n = \langle\widetilde{\nabla}_{e_2} e_2, N\rangle = \frac{-yv' + v''}{\sqrt{1 + y^2 + v'^2}}.$$

Let K_{ext} be the extrinsic Gauss curvature of Σ,

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2}{(1 + x^2 + f'^2)^2}. \tag{4.1}$$

Thus Σ is extrinsically flat invariant surface in Heisenberg group $\mathbb{H}_3$ if and only if

$$K_{ext} = 0,$$

that is, if and only if

$$x^2 + xf'f'' - \frac{1}{4}(x^2 + f'^2 + 1)^2 = 0. \tag{4.2}$$

to classify extrinsically flat invariant surfaces must solve the equation (4.2).

We remark that the equation (4.2) is similarly to the equation (3.1), It is sufficient to change y by x and v by f.

As conclusion, we have

Theorem 4.1. $\bullet$*The only non-extendable extrinsically flat translation surfaces in the 3-dimensional Heisenberg group* $\mathbb{H}_3$ *invariant under the* 2*-parameter subgroup* $G_2 = \{(0, t, 0) \in \mathbb{H}_3/t \in \mathbb{R}\}$, *are the surfaces whose parametrization is* $X(x, y) = \left(x, y, f(x) - \frac{xy}{2}\right)$ *where* x *and* f *satisfy*

$$f(x) = \int \sqrt{\frac{2}{-\ln(x) + \alpha} - x^2 - 1dy}.$$

where $\alpha \in \mathbb{R}$, *and* $x \neq e^{\alpha}$.

$\bullet$*There are no complete extrinsically flat translation surfaces in the 3-dimensional Heisenberg group* $\mathbb{H}_3$ *invariant under the 1-parameter subgroup* $G_2 = \{(0, t, 0) \in \mathbb{H}_3/t \in \mathbb{R}\}$.

Remark 4.2. Let Σ be a G_2-invariant translation surfaces in the 3-dimensional Heisenberg space. Then Σ is locally expressed as

$$X(x, y) = (x, 0, f(x)) \,.\, (0, y, 0) = \left(x, y, f(x) + \frac{xy}{2}\right).$$

Then the extrinsically Gaussian curvature K_{ext} of Σ is computed as

$$K_{ext} = -\frac{\left((f' + y)^2 - 1\right)^2}{4\left(1 + (v' - x)^2\right)^2}.$$

Thus Σ can not be of constant extrinsically Gaussian curvature.

Theorem 4.3. $\bullet$ *The* G_2*-invariant constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group* $\mathbb{H}_3$, *are:*

1. $K_{ext} = -\frac{1}{4}$.
   The surfaces of equation

$$z = f(x) - \frac{xy}{2} = -\frac{xy}{2} + \frac{1}{2}x\sqrt{2\beta - x^2} + \arctan\left(\frac{x}{\sqrt{\beta - x^2}}\right),$$

   where $\beta \in \mathbb{R}$.

2. $K_{ext} \neq -\frac{1}{4}$.
   Then x *and* f *satisfy*

$$f(x) = \int \sqrt{\frac{1}{-2(K_{ext} + \frac{1}{4})\ln(x) + \gamma} - x^2 - 1}dy.$$

   where $\gamma \in \mathbb{R}$, *and* $x \neq e^{\frac{\gamma}{2(K_{ext}+\frac{1}{4})}}$.

• *There are no complete constant extrinsically Gaussian curvature translation surfaces in the 3-dimensional Heisenberg group* $\mathbb{H}_3$ *invariant under the 1-parameter subgroup* G_2.

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